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Numerical methods for solving singular integral equations with Cauchy-type kernels

A Mathunyane, K Motjeane, L Hlophe, K Moipolai and M Nchabeleng (Problem presenter)

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Overview

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Introduction and background

Introduction and background

Singular Integral Equations

- Singular integral equations of the first kind with Cauchy-type kernels are a class of mathematical equations that arise in the field of integral equations.
- The general form of a singular integral equation of the first kind with a Cauchy-type kernel is given by:

$$\int_a^b \frac{\varphi(t)}{t-x}\mathrm{d}t = f(x), \quad a < x < b, \tag{1}$$

  where the forcing function $f(x)$ is given and the function $\varphi(t)$ is the unknown function to be determined.
- The singularity in the kernel occurs when x is equal to t, leading to challenges in the analysis and solution of these equations.

Introduction and background

Singular Integral Equations

- Singular integral equations play a crucial role in various branches of applied mathematics and physics, such as potential theory, elasticity, and fluid dynamics.
- They often arise in problems involving boundary value conditions and are used to model physical phenomena in diverse areas.
- The study of singular integral equations involves techniques from functional analysis, complex analysis, and integral transforms.
- Solving these equations can be challenging due to their singular nature, and various methods, such as regularization techniques, numerical methods, and special function expansions, are often employed to obtain solutions.

Problem statement

Problem statement

Cauchy-type singular integral equation

- Consider the problem of solving the singular integral equation given by [1]

$$\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^1 \frac{\varphi_\xi(\xi)}{\xi - x}\mathrm{d}\xi\right) = 1, \qquad 0 \leqslant x \leqslant 1, \tag{2}$$

subject to the boundary conditions

$$\varphi_x(0) = 0, \quad \varphi(1) = 0. \tag{3a-b}$$

- We want to solve (2) subject to (3a-b) both analytically and numerically.

Analytical solution

Analytical solution

- We begin by integrating equation (2) with respect to x, resulting in the derivation of the characteristic singular integral equation:

$$\int_0^1 \frac{\varphi_\xi(\xi)}{\xi - x} \mathrm{d}\xi = \pi x + A. \tag{4}$$

where A is an arbitrary constant.

- Using the standard inversion formula [2], we obtain the inverse of (4) as

$$\varphi_x(x) = \frac{C}{\sqrt{x(1-x)}} - \frac{1}{\pi^2\sqrt{x(1-x)}} \int_0^1 \frac{\sqrt{\xi(1-\xi)}(\pi\xi + A)}{\xi - x} \mathrm{d}\xi. \tag{5}$$

Analytical solution

- When evaluated
$$\int_0^1 \frac{\sqrt{\xi(1-\xi)}(\pi\xi + A)}{\xi - x}\mathrm{d}\xi = \frac{\pi}{8}\left(1 + 4x - 8x^2\right) + \frac{A\pi}{2}(1-2x)\,. \qquad (6)$$
- Integration of (5) yields, given (6)
$$\varphi(x) = 2C\sin^{-1}\left(\sqrt{x}\right) + \left(\frac{x}{2} + \frac{1}{4} + \frac{A}{\pi}\right)\sqrt{x(1-x)} + B, \qquad (7)$$
  where A, B and C are unknown constants.
- The solution in (7) contains three unknown constants. We will therefore need three conditions to solve for the three unknowns. We will now assume
$$\varphi(0) = \frac{3\pi}{8}. \qquad (8)$$

Analytical solution

- Using the boundary conditions (2) and (8), we find

$$A = -\pi,\ \ B = \frac{3\pi}{16} \text{ and } C = -\frac{3}{8}. \tag{9}$$

- Therefore, the analytical solution is given by

$$\varphi(x) = -\frac{3}{4}\sin^{-1}\left(\sqrt{x}\right) + \left(\frac{3}{4} - \frac{x}{2}\right)\sqrt{x(1-x)} + \frac{3\pi}{8}. \tag{10}$$

Numerical techniques and solutions

Numerical method

A conventional numerical discretisation

- We divide the interval $[0, 1]$ into n-equally spaced sub-intervals $[\xi_j, \xi_{j+1}]$ of length $h = 1/n$ where $0 \leqslant j \leqslant n-1$.
- Let $P(x)$ be defined as

$$P(x) = \int_0^1 \frac{\varphi_\xi(\xi)}{\xi - x} \mathrm{d}\xi, \tag{11}$$

then (1) becomes

$$\frac{dP}{dx} = \pi. \tag{12}$$

Numerical method

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- Using the central finite difference approximations to approximate $\mathrm{d}P/\mathrm{d}x$ and evaluating P at the mid-grid points where it can be evaluated, equation (12) becomes

$$\frac{P_{i+1/2} - P_{i-1/2}}{\xi_{i+1/2} - \xi_{i-1/2}} = \pi, \quad 1 \leqslant i \leqslant n-1, \tag{13}$$

where $P_{i\pm1/2}$ and $\xi_{i\pm1/2}$ are given by

$$P_{i\pm1/2} = \int_0^1 \frac{\varphi_\xi(\xi)}{\xi - \xi_{i\pm1/2}} \mathrm{d}\xi \text{ and } \xi_{i\pm1/2} = \frac{i \pm 1/2}{n}, \tag{14}$$

respectively.

Numerical method

- Assuming that the slope is constant in each sub-interval and by approximating $\varphi_\xi(\xi)$ using forward differences, we obtain

$$\sum_{j=0}^{n-1}\left(\frac{\varphi_{j+1}-\varphi_j}{\xi_{j+1}-\xi_j}\right)\left(\int_{\xi_j}^{\xi_{j+1}}\frac{1}{\xi-\xi_{i+1/2}}\mathrm{d}\xi-\int_{\xi_j}^{\xi_{j+1}}\frac{1}{\xi-\xi_{i-1/2}}\mathrm{d}\xi\right)=\frac{\pi}{n}. \tag{15}$$

- When evaluated

$$\int_{\xi_j}^{\xi_{j+1}}\frac{\mathrm{d}\xi}{\xi-\xi_{i\pm1/2}}=\ln\left|\frac{\xi_{j+1}-\xi_{i\pm1/2}}{\xi_{j+1}-\xi_{i\pm1/2}}\right|. \tag{16}$$

- Equation (15) becomes, using (16)

$$\sum_{j=0}^{n-1}\left(\varphi_{j+1}-\varphi_j\right)a_{ij}=\frac{\pi}{n^2}, \tag{17}$$

where $\xi_{i+1/2}-\xi_{i-1/2}=\xi_{j+1}-\xi_j=1/n$ and

Numerical method

$$a_{i,j} = \ln\left|\frac{(2j-2i+1)^2}{(2j-2i+3)(2j-2i-1)}\right|. \tag{18}$$

- Expanding the summation and evaluating the resulting equation at $i = 1, 2, ..., n-1$ generates a system of $n-1$ linear equations in $n+1$ unknowns. Imposing the boundary condition $\varphi(x_0) = \varphi_0, \varphi_x(0) = 0$ and $\varphi(x_n) = 0$, we are able to determine three unknown constants to get an $n-1$ system in $n-2$ unknowns.
- Consequently, we have an over-determined system. Any $n-2$ equations are therefore sufficient to determine the remaining unknowns. To demonstrate this idea, we set $n = 5$.

Numerical method

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- Equation (17) becomes

$$\sum_{j=0}^{4} \left(\varphi_{j+1} - \varphi_j\right) a_{ij} = \frac{\pi}{25}, \quad 1 \leqslant i \leqslant 4. \tag{19}$$

- Expanding the summation and collecting φ terms, we get

$$-a_{i0}\varphi_0+(a_{i0}-a_{i1})\varphi_1+(a_{i1}-a_{i2})\varphi_2+(a_{i2}-a_{i3})\varphi_3+(a_{i3}-a_{i4})\varphi_4+a_{i4}\varphi_5 = \frac{\pi}{25}. \tag{20}$$

- From the boundary conditions:

$$\varphi_x(0) = 0 \implies n(\varphi_1 - \varphi_0) = 0 \implies \varphi_1 = \varphi_0 \tag{21}$$

$$\varphi(1) = 0 \implies \varphi_5 = \varphi(1) = 0. \tag{22}$$

- Assuming that $\varphi(0) = \frac{3\pi}{8}$ implies that $\varphi_1 = \varphi_0 = \frac{3\pi}{8}$.

Numerical method

- Equation (20) becomes

$$(a_{i1} - a_{i2})\varphi_2 + (a_{i2} - a_{i3})\varphi_3 + (a_{i3} - a_{i4})\varphi_4 = \frac{\pi}{25} + a_{i1}\frac{3\pi}{8}, \quad 1 \leqslant i \leqslant 4. \tag{23}$$

- When evaluated (23) generates the linear system given by

$$(a_{11} - a_{12})\varphi_2 + (a_{12} - a_{13})\varphi_3 + (a_{13} - a_{14})\varphi_4 = \frac{\pi}{25} + \frac{3\pi}{8}a_{11}, \tag{24}$$

$$(a_{21} - a_{22})\varphi_2 + (a_{22} - a_{23})\varphi_3 + (a_{23} - a_{24})\varphi_4 = \frac{\pi}{25} + \frac{3\pi}{8}a_{21}, \tag{25}$$

$$(a_{31} - a_{32})\varphi_2 + (a_{32} - a_{33})\varphi_3 + (a_{33} - a_{34})\varphi_4 = \frac{\pi}{25} + \frac{3\pi}{8}a_{31}, \tag{26}$$

$$(a_{41} - a_{42})\varphi_2 + (a_{42} - a_{43})\varphi_3 + (a_{43} - a_{44})\varphi_4 = \frac{\pi}{25} + \frac{3\pi}{8}a_{41}. \tag{27}$$

Numerical results

Graph of $\varphi(x)$ plotted against x when $n = 5$.

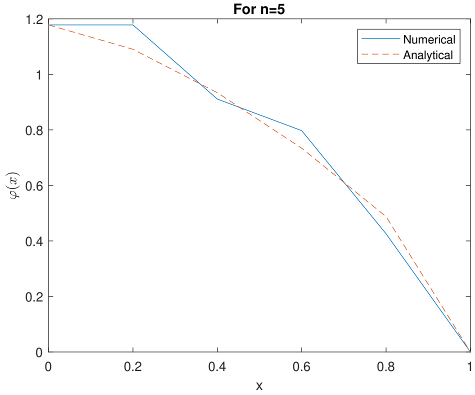

Figure 1: Graph of $\varphi(x)$ when $1 \leqslant i \leqslant 3$.

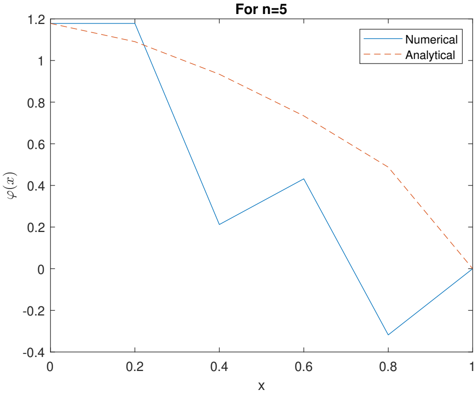

Figure 2: Graph of $\varphi(x)$ when $2 \leqslant i \leqslant 4$.

Numerical results

Graph of $\varphi(x)$ plotted against x when $n = 100$.

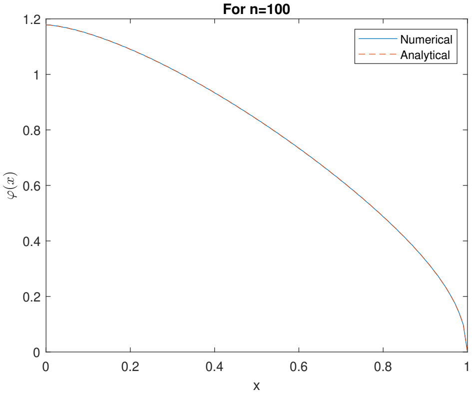

Figure 3: Graph of $\varphi(x)$ when $1 \leqslant i \leqslant 98$.

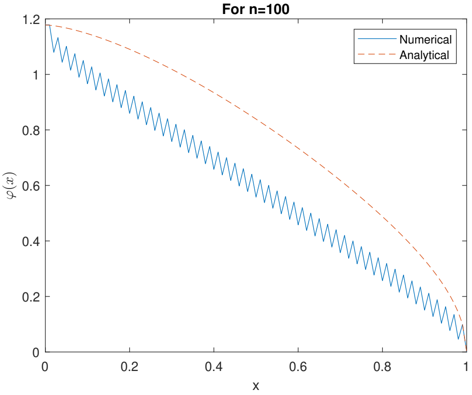

Figure 4: Graph of $\varphi(x)$ when $2 \leqslant i \leqslant 99$.

Numerical results

Relative error for $n = 100$.

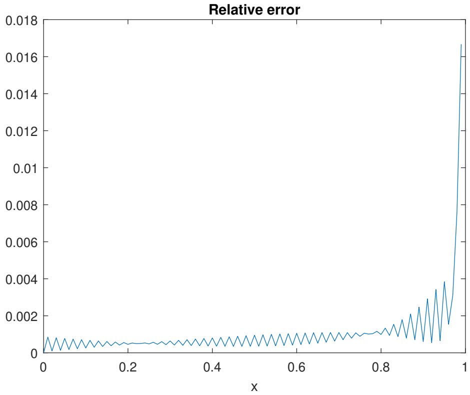

Figure 5: Relative error when $1 \leqslant i \leqslant 98$.

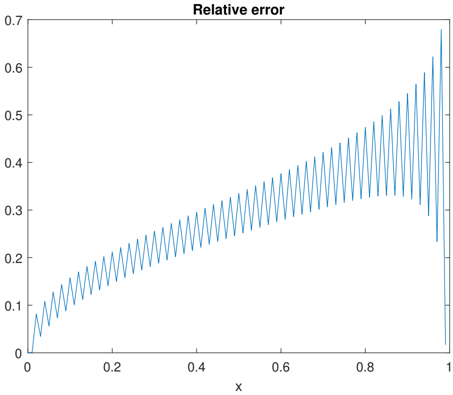

Figure 6: Relative error when $2 \leqslant i \leqslant 99$.

Numerical method

Problem reformulation and regularisation

- It is clear from the analytical solution (10) that $\varphi(x)$ approaches zero like $\sqrt{1-x}$ as $x \to 1$. Consequently, a singularity emerges in the slope of $\varphi(x)$ at $x = 1$, that is, $\varphi_x(x) \to \infty$ as $x \to 1$.
- This singularity at $x = 1$ poses a challenge to the numerical scheme employed in the preceding section. To overcome this challenge, we introduce the following transformation $\varphi(x) = h(y), \;$ where $\; y = \sqrt{1-x}$.
- Under these circumstances, (1) becomes

$$\frac{1}{\pi}\frac{\mathrm{d}}{\mathrm{d}y}\left(\int_0^1 \frac{h_\xi(\xi)}{\xi - x}\mathrm{d}\xi\right) = 1, \qquad 0 \leqslant x \leqslant 1, \tag{28}$$

subject to the boundary conditions $h(0) = 0$, $h_y(1) = 0$ and $h(1) = 3\pi/8$.

Numerical method

- Let
$$P = \int_0^1 \frac{h_\eta(\eta)\mathrm{d}\eta}{y^2 - \eta^2}, \tag{29}$$
so that
$$\frac{dP}{dy} = 2\pi y. \tag{30}$$
- Then, the central finite differences can be used to approximate P_y to get
$$\frac{P_{i+1/2} - P_{i-1/2}}{\eta_{i+1/2} - \eta_{i-1/2}} = \frac{2\pi i}{n}, \tag{31}$$
where $P_{i\pm 1/2}$ and $\eta_{i\pm 1/2}$ are respectively given by
$$P_{i\pm 1/2} = \int_0^1 \frac{h_\eta(\eta)\mathrm{d}\eta}{\eta_{i\pm 1/2}^2 - \eta^2}, \quad \text{and} \quad \eta_{i\pm 1/2} = \frac{i \pm 1/2}{n}. \tag{32a-b}$$

Numerical method

- Assuming that the slope $\mathrm{d}h/\mathrm{d}\eta$ is constant in each sub-interval and approximating $h_\eta(\eta)$ using forward differences, we obtain

$$P_{i\pm1/2} = \sum_{j=0}^{n-1}\left(\frac{h_{j+1}-h_j}{\eta_{j+1}-\eta_j}\right)\int_{\eta_j}^{\eta_{j+1}}\frac{\mathrm{d}\eta}{\eta_{i\pm1/2}^2-\eta^2}. \tag{33}$$

- When evaluated

$$\int_{\eta_i}^{\eta_{i+1}}\frac{\mathrm{d}\eta}{\eta_{i\pm1/2}^2-\eta^2} = \frac{1}{2\eta_{i\pm1/2}}\ln\left|\frac{\left(\eta_{i\pm1/2}-\eta_j\right)\left(\eta_{i\pm1/2}+\eta_{j+1}\right)}{\left(\eta_{i\pm1/2}+\eta_j\right)\left(\eta_{i\pm1/2}-\eta_{j+1}\right)}\right|. \tag{34}$$

Numerical method

- Substituting (33) into (31) we obtain, using (34)

$$\sum_{j=0}^{n-1}(h_{j+1} - h_j)b_{ij} = \frac{2\pi i}{n^4}, \tag{35}$$

  where

$$b_{ij} = \frac{1}{2i+1}\ln\left|\frac{(2i+2j+3)(2i-2j+1)}{(2i+2j+1)(2i-2j-1)}\right| - \frac{1}{2i-1}\ln\left|\frac{(2i+2j+1)(2i-2j-1)}{(2i+2j-1)(2i-2j-3)}\right|. \tag{36}$$

- When evaluated for any value of n, equation (35) will generate a system of $n-1$ linear equations in $n-2$ unknowns. Since the transformation removed the singularity in the slope of $h(x)$ at $x = 1$, any combination of $n-2$ equations can be used to solve the system.

Numerical results

Graph of $\varphi(x)$ plotted against x when $n = 5$.

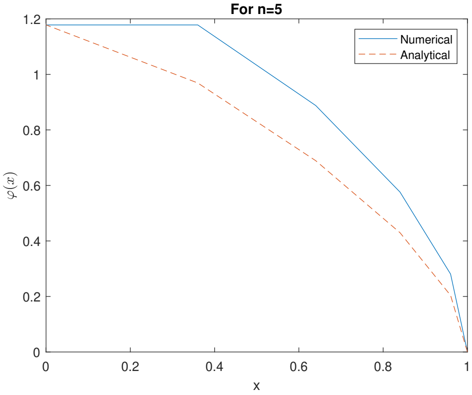

Figure 7: Graph of $\varphi(x)$ when $1 \leqslant i \leqslant 3$.

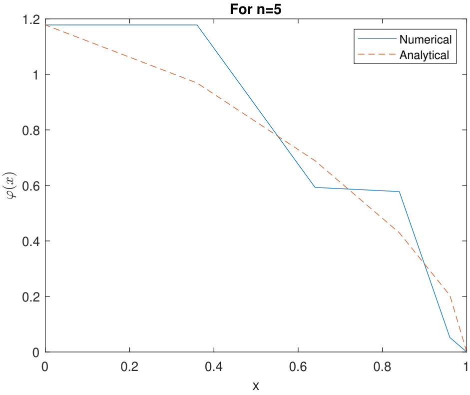

Figure 8: Graph of $\varphi(x)$ when $2 \leqslant i \leqslant 4$.

Numerical results

Graph of $\varphi(x)$ plotted against x when $n = 100$.

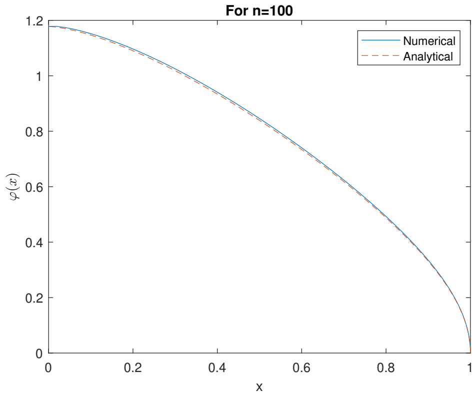

Figure 9: Graph of $\varphi(x)$ when $1 \leqslant i \leqslant 98$.

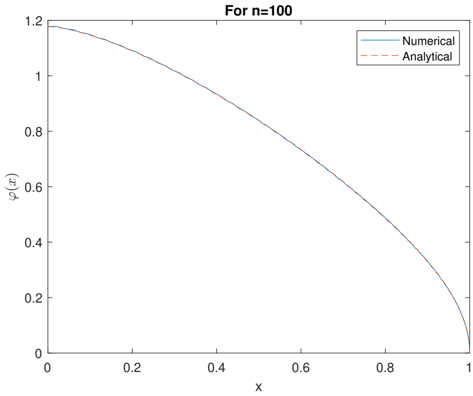

Figure 10: Graph of $\varphi(x)$ when $2 \leqslant i \leqslant 99$.

Numerical results

Relative error for $n = 100$.

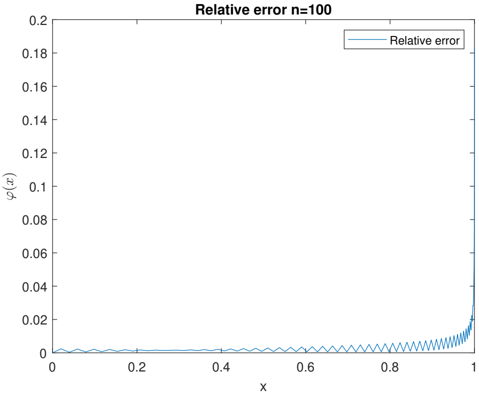

Figure 11: Relative error when $1 \leqslant i \leqslant 98$.

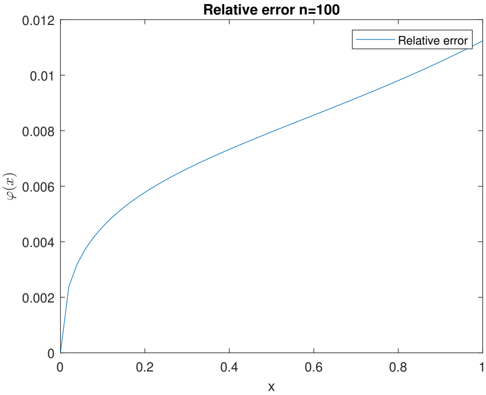

Figure 12: Relative error when $2 \leqslant i \leqslant 99$.

Conclusion

Conclusion

- This study focused on exploring numerical solutions for Cauchy-type singular integral equations of the first kind.
- Two distinct numerical techniques were employed to compute solutions, and the obtained results were compared with the analytical solution to assess the accuracy of these methods.
- Relative error plots from the numerical approaches were generated and analyzed.
- A notable challenge emerged due to the singularity in $\varphi_x(x)$ at $x = 1$, which disrupted the conventional finite difference scheme.
- To address this challenge, we opt for an alternative approach by solving a regularized problem.

QUESTIONS?

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